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Solving minimax problems with feasible sequential quadratic programming

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Background

- ▶ Feasible sequential quadratic programming (FSQP) refers to a class of sequential quadratic programming methods.
- ▶ Engineering applications: number of variables is not large; evaluations of objective or constraint functions and of the gradients are time consuming
- ▶ Advantages:
 - ▶ 1) Generating feasible iterates.
 - ▶ 2) Reducing the amount of computation.
 - ▶ 3) Enjoying the same global and fast local convergence properties.

Background

- ▶ The constrained mini-max problem

Minimize $\max_{i \in I'} \{f_i(x)\}$ s.t. $x \in X$

X is the set of points $x \in \mathbb{R}^n$ satisfying

$$\begin{cases} bl \leq x \leq bu & * \text{ Bounds} \\ g_j(x) \leq 0, & j = 1, \dots, n_i \\ g_j(x) \equiv \langle c_{j-n_i}, x \rangle - d_{j-n_i} \leq 0, & j = n_i + 1, \dots, t_i \\ h_j(x) = 0, & j = 1, \dots, n_e \\ h_j(x) \equiv \langle a_{j-n_e}, x \rangle - b_{j-n_e} = 0, & j = n_e + 1, \dots, t_e \end{cases}$$

* Nonlinear inequality

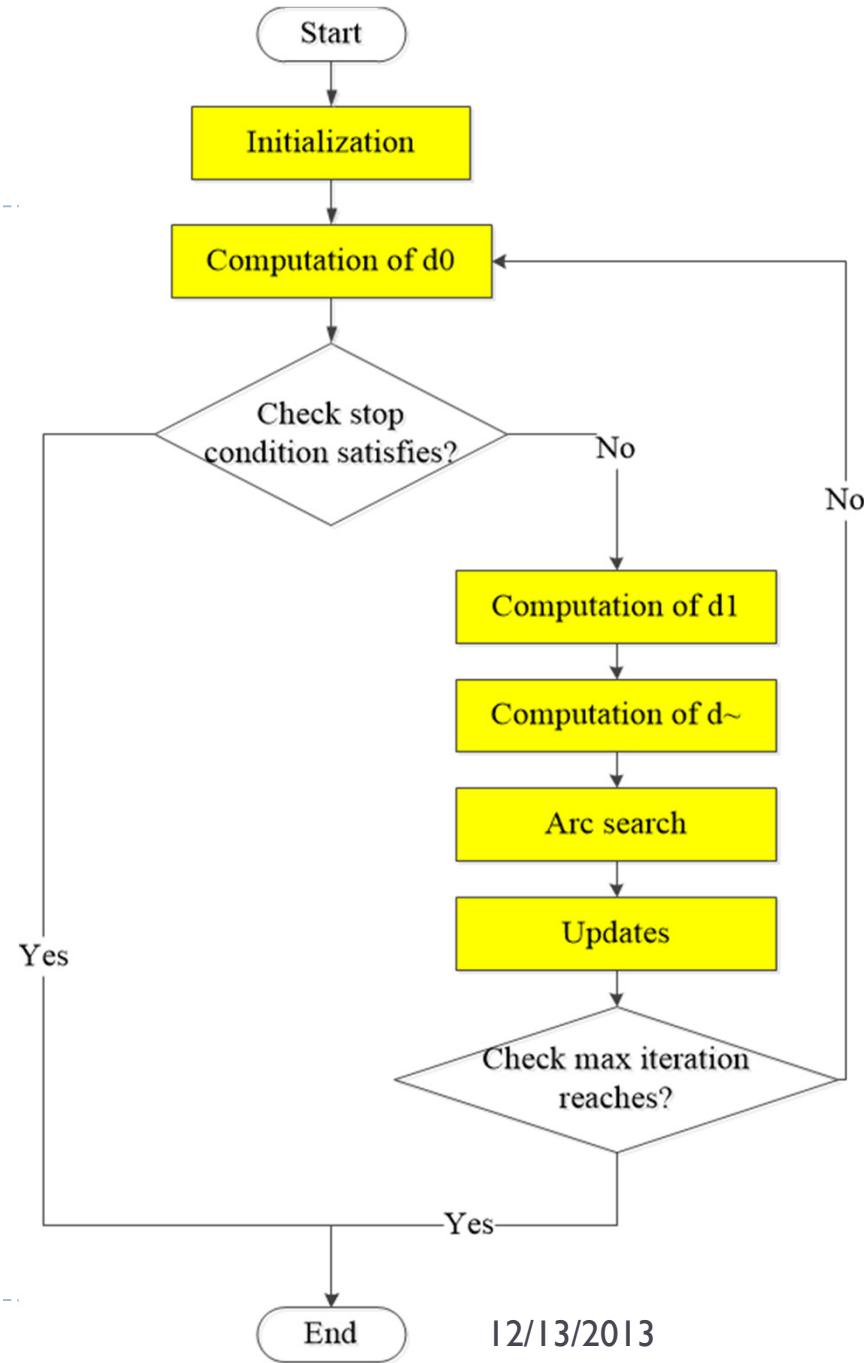
* Linear inequality

* Nonlinear equality

* Linear equality

Algorithm - FSQP

- * Step 1. Initialization
- * Step 2. A search arc
- * Step 3. Arc search
- * Step 4. Updates



Initialization (x, H, p, k)

A feasible point x_0

Initial guess x_{00}

Infeasible for linear constraints	Strictly convex quadratic program
Infeasible for the nonlinear inequality constraints	Armijo-type line search

Initial Hessian matrix H_0 = the identity matrix,

$p_{0,j} = \varepsilon_2$ for $j = 1, \dots, n_e$ Positive penalty parameters $f_m(x, p) = \max_{i \in I^f} \{f_i(x)\} - \sum_{j=1}^{n_e} p_j h_j(x)$

Iteration index $k = 0$

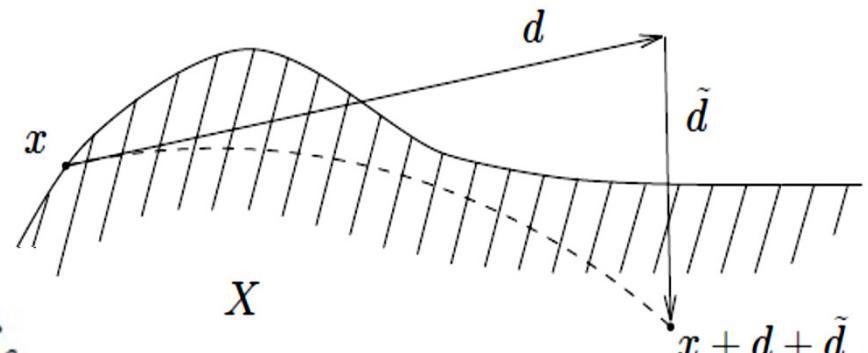
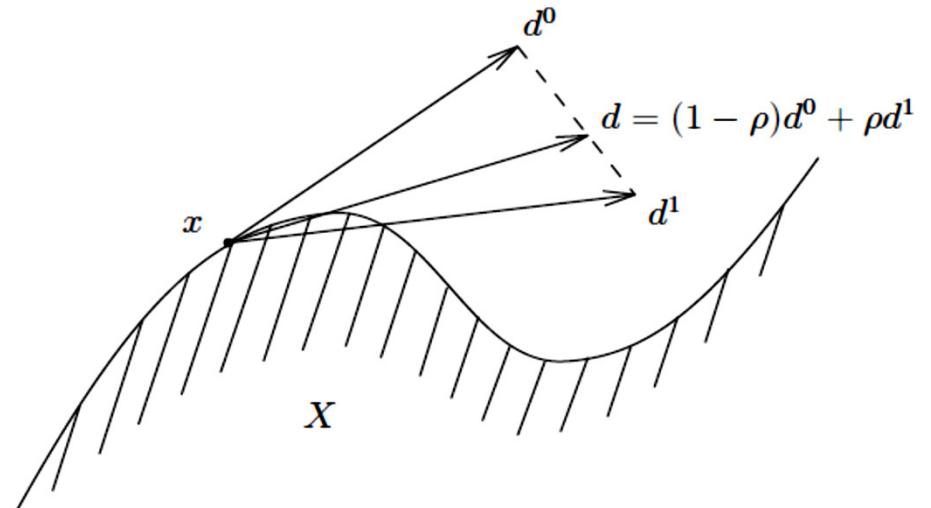
Computation of a search arc

1. Compute d_k^0

2. Compute d_k^1

3. Set $d_k = (1 - \rho_k)d_k^0 + \rho_k d_k^1$

4. Compute \tilde{d}_k



Stop: if $\|d_k^0\| \leq \varepsilon$ and $\sum_{j=1}^{n_e} |h_j(x_k)| \leq \varepsilon_e$

Computation of a search arc

1. Compute d_k^0 , solution of $QP(x_k, H_k, p_k)$

$$\begin{cases} \min_{d^0} & \frac{1}{2} \langle d^0, H_k d^0 \rangle + f'(x_k, d^0, p_k) \\ s.t. & bl \leq x_k + d^0 \leq bu \\ & g_j(x_k) + \langle \nabla g_j(x_k), d^0 \rangle \leq 0, \quad j = 1, \dots, t_i \\ & h_j(x_k) + \langle \nabla h_j(x_k), d^0 \rangle \leq 0, \quad j = 1, \dots, n_e \\ & \langle a_j, x_k + d^0 \rangle = b_j \quad j = 1, \dots, t_e - n_e \end{cases}$$

$$f'(x, d, p) = \max_{i \in I^f} \{f_i(x) + \langle \nabla f_i(x), d \rangle\} - f_{I^f}(x) - \sum_{j=1}^{n_e} p_j \langle \nabla h_j(x), d \rangle$$

$$f_I(x) = \max_{i \in I} \{f_i(x)\}$$

2. Compute d_k^1 , solution of $QP(x_k, d_k^0, p_k)$

$$\begin{cases} \min_{d^1 \in \mathbb{R}^n, \gamma \in \mathbb{R}} & \frac{\eta}{2} \langle d_k^0 - d^1, d_k^0 - d^1 \rangle + \gamma \\ s.t. & bl \leq x_k + d^1 \leq bu \\ & f'(x_k, d^1, p_k) \leq \gamma \\ & g_j(x_k) + \langle \nabla g_j(x_k), d^1 \rangle \leq \gamma \quad j = 1, \dots, n_i \\ & \langle c_j, x_k + d^1 \rangle \leq d_j \quad j = 1, \dots, t_i - n_i \\ & h_j(x_k) + \langle \nabla h_j(x_k), d^1 \rangle \leq \gamma \quad j = 1, \dots, n_e \\ & \langle a_j, x_k + d^1 \rangle = b_j \quad j = 1, \dots, t_e - n_e \end{cases}$$

3. Set $d_k = (1 - \rho_k) d_k^0 + \rho_k d_k^1$

$$\rho_k = \|d_k^0\|^\kappa / (\|d_k^0\|^\kappa + v_k)$$

$$v_k = \max(0.5, \|d_k^1\|^{\tau_1})$$

4. Compute \tilde{d}_k

Quadratic programming

Strictly convex quadratic programming:

- Unique global minimum
- Matrix C need to be positive definite

$$\text{Minimize } \frac{1}{2} X' C X + D' X$$

m = number of constraints;

Subject to $A X \leq B$

n = number of variables

$$X \geq 0$$

$$X_{n \times 1} = [x_1, x_2, \dots, x_n]'$$

$$A_{m \times n}; B_{m \times 1}; C_{n \times n}; D_{n \times 1}$$

Extended Wolfe's simplex method

- No derivative

Quadratic programming

Lagrangian function: $L(x, \lambda) = \frac{1}{2} X' CX + D' X + \lambda(AX - B)$

Karush-Kuhn-Tucker conditions:

$$\left\{ \begin{array}{l} \frac{\partial L}{\partial \lambda} = AX - B \leq 0 \\ \frac{\partial L}{\partial X} = X' C + D' + \lambda A \geq 0; CX + D + A' \lambda' \geq 0 \\ \lambda \frac{\partial L}{\partial \lambda} = \lambda(AX - B) = 0 \\ X' \frac{\partial L}{\partial X} = X'(CX + D + A' \lambda') = 0 \end{array} \right. \xrightarrow{\hspace{1cm}} \left\{ \begin{array}{l} AX + v = B \\ CX + A' \lambda' - \mu + s = -D \\ \lambda v = 0 \\ \mu' X = 0 \\ X \geq 0, \lambda \geq 0, \mu \geq 0, v \geq 0, s \geq 0 \end{array} \right.$$

v = slack variables; μ = surplus variables

s = artificial variables

(λ, v) and (μ', X) = complementary slack variables

Linear programming

Quadratic programming: => **Conditioned linear programming:**

$$\text{Minimize } \frac{1}{2} X' CX + D' X$$

Subject to $AX \leq B$

$$X \geq 0$$

$$\text{Minimize } M' \cdot s$$

$$\text{Subject to } AX + v = B$$

$$CX + A'\lambda' - \mu + s = -D$$

$$X, \lambda, \mu, v, s \geq 0$$

(λ, v) and (μ, X) are complementary slack variables.

Simplex tableau:

		1	n	m	n	m	n
		rhs	X	λ'	μ	v	s
1	Z						M'
	v	B	A			I	
	s	- D	C	A'	- I		I

v and s are basis variables

Test example

Minimize $f(x) = x_1^2 + 2x_2^2 - 2x_1x_2 - 4x_1$

Subject to

$$2x_1 + x_2 \leq 6.0$$

$$x_1 - 4x_2 \leq 0.0$$

$$x_1, x_2 \geq 0.0$$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, A = \begin{bmatrix} 2 & 1 \\ 1 & -4 \end{bmatrix}, B = \begin{bmatrix} 6 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}, D = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

$m = 2$ number of constraints; $n = 2$ number of variables

Optimal solution

$$x_1^* = 2.462, x_2^* = 1.077, f(x^*) = -6.769$$

Arc search

$$\delta_k = f'(x_k, d_k, p_k) \quad \text{If } n_i + n_e \neq 0 \text{ and } \delta_k = -\langle d_k^0, H_k d_k^0 \rangle.$$

Compute t_k , the first number t in the sequence $\{1, \beta, \beta^2, \dots\}$ satisfying

$$f_m(x_k + t d_k + t^2 \tilde{d}_k, p_k) \leq f_m(x_k, p_k) + \alpha t \delta_k$$

$$g_j(x_k + t d_k + t^2 \tilde{d}_k) \leq 0, \quad j = 1, \dots, n_i.$$

$$\langle c_{j-n_i}, x_k + t d_k + t^2 \tilde{d}_k \rangle \leq d_{j-n_i}, \quad \forall j > n_i \text{ and } j \notin I_k^g(d_k)$$

$$h_j(x_k + t d_k + t^2 \tilde{d}_k) \leq 0, \quad j = 1, \dots, n_e.$$

Updates (x, H, p, k)

$$x_{k+1} = x_k + t_k d_k + t_k^2 \tilde{d}_k$$

BFGS formula to compute $H_{k+1} = H_{k+1}^T > 0$.

$$p_{k+1,j} = \begin{cases} p_{k,j} & p_{k,j} + \bar{\mu}_j \geq \epsilon_1 \\ \max\{\epsilon_1 - \bar{\mu}_j, \delta p_{k,j}\} & otherwise \end{cases}$$

Solve the unconstrained quadratic problem in $\bar{\mu}$

$$\min_{p \in \mathbb{R}^{n_k}} \left\| \sum_{j=1}^{n_f} \zeta_{k,j} \nabla f_j(x_{k+1}) + \zeta_k + \sum_{j=1}^{t_i} \lambda_{k,j} \nabla g_j(x_{k+1}) + \sum_{j=n_e+1}^{t_e} \mu_{k,j} \nabla h_j(x_{k+1}) + \sum_{j=1}^{n_k} \bar{\mu}_j \nabla h_j(x_{k+1}) \right\|^2$$

$$k = k + 1$$

Updates

BFGS formula with Powell's modification:

H_{k+1} is the Hessian of the Lagrangian function $f_m(x_k, p_k)$

$$\boldsymbol{\eta}_{k+1} = \boldsymbol{x}_{k+1} - \boldsymbol{x}_k$$

$$\boldsymbol{\gamma}_{k+1} = \nabla_x f_m(\boldsymbol{x}_{k+1}, \boldsymbol{p}_k) - \nabla_x f_m(\boldsymbol{x}_k, \boldsymbol{p}_k)$$

$$\boldsymbol{\xi}_{k+1} = \boldsymbol{\theta}_{k+1} \cdot \boldsymbol{\gamma}_{k+1} + (1 - \boldsymbol{\theta}_{k+1}) \cdot \boldsymbol{H}_k \boldsymbol{\delta}_{k+1} \quad \boldsymbol{\theta}_{k+1} = \begin{cases} 1, & \boldsymbol{\eta}_{k+1}^T \boldsymbol{\gamma}_{k+1} \geq 0.2 \boldsymbol{\eta}_{k+1}^T \boldsymbol{H}_k \boldsymbol{\eta}_{k+1} \\ \frac{0.8 \boldsymbol{\eta}_{k+1}^T \boldsymbol{H}_k \boldsymbol{\eta}_{k+1}}{\boldsymbol{\eta}_{k+1}^T \boldsymbol{H}_k \boldsymbol{\eta}_{k+1} - \boldsymbol{\eta}_{k+1}^T \boldsymbol{\gamma}_{k+1}} & \text{otherwise} \end{cases}$$

$$\boldsymbol{H}_{k+1} = \boldsymbol{H}_k - \frac{\boldsymbol{H}_k \boldsymbol{\eta}_{k+1} \boldsymbol{\eta}_{k+1}^T \boldsymbol{H}_k}{\boldsymbol{\eta}_{k+1}^T \boldsymbol{H}_k \boldsymbol{\eta}_{k+1}} + \frac{\boldsymbol{\xi}_{k+1} \boldsymbol{\xi}_{k+1}^T}{\boldsymbol{\eta}_{k+1}^T \boldsymbol{\xi}_{k+1}}$$

Project Schedule

October	<ul style="list-style-type: none">• Literature review;• Specify the implementation module details;• Structure the implementation;
November	<ul style="list-style-type: none">• Develop the quadratic programming module;• Unconstrained quadratic program;• Strictly convex quadratic program;• Validate the quadratic programming module;
December	<ul style="list-style-type: none">• Develop the Gradient and Hessian matrix calculation module;• Validate the Gradient and Hessian matrix calculation module;• Midterm project report and presentation;

January	<ul style="list-style-type: none"> • Develop Armijo line search module; • Validate Armijo line search module;
February	<ul style="list-style-type: none"> • Develop the feasible initial point module; • Validate the feasible initial point module; • Integrate the program;
March	<ul style="list-style-type: none"> • Debug and document the program; • Validate and test the program with case application;
April	<ul style="list-style-type: none"> • Add arch search variable \tilde{d} in; • Compare calculation efficiency of line search with arch search methods;
May	<ul style="list-style-type: none"> • Develop the user interface if time available; • Final project report and presentation;

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